RANK CHARACTERS FOR GENERALIZED PERSISTENCE MODULES

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Abstract. It's well known that the rank of its internal linear transformations alone is not sufficient to determine the isomorphism type of a multiparameter persistence module. In this paper, we use the rank of matrices to define *rank characters* for the generalized persistence modules of an arbitrary finite poset. Using rank characters, we assign to a persistence module its *virtual barcode*, a vector over $\mathbb{Q}$, indexed by the collection of convex (interval) modules. This invariant, which can be computed explicitly, conveys information about the summands of a persistence module M. For example, when a module's virtual barcode has a coordinate that is not in $\mathbb{Z}_{\geq 0}$, the module cannot decompose into a direct sum of convex modules. On the other hand, if M is a direct sum of convex modules, its virtual barcode corresponds exactly to its Krull-Schmidt decomposition. In this case, the barcode of M can be obtained from its rank character by matrix multiplication.

1. Introduction

A subset S of a poset P is **convex**[1], if it's edge connected and if $[a, b] \subseteq S$ whenever $a, b \in S$. It's well known that the convex subsets of P parametrize the convex modules for P. That is, the isomophism classes of those generalized persistence modules which are indecomposable and determined completely by their support are supported on convex subsets of P. While rank can be used as an invariant for one-dimensional persistence modules in the sense that rank determines the isomorphism type (see [CSEH07], [CdGO12]), when a poset isn't totally ordered the ranks of the matrices corresponding to the intervals in the poset aren't sufficient to distinguish even between direct sums of convex modules; see Example 1.1 below. In a sense, the problem is precisely that there are convex modules with supports that *do not correspond to intervals in the poset at all.* This issue is significant, since the only connected posets with the property that all convex modules are intervals are totally ordered. Thus, with an eye on multiparameter persistence, a better way of using rank is required.

Example 1.1. We consider the poset P with its representations M and N (see Figure 1). Note that both M and N agree on every *interval* in P with respect to rank. That is,

$$\mathrm{Rank}(M(x)) = \mathrm{Rank}(N(x)) = \begin{cases} 2, & x = [1,1] \\ 1, & x = [2,2], [3,3], [1,3], [2,3] \\ 0, & x = [1,4], [2,4], [3,4], [4,4]. \end{cases}$$

Note that both M and N are indeed direct sums of convex modules, yet the rank of the matrices for all their intervals alone does not distinguish them. In Example 4.1, we use rank characters and virtual barcodes to separate M and N. □

Recent work has suggested criteria for determining when a multiparameter persistence module is a direct sum of certain types of convex modules (see [CO16], [BLO20]). Others have explored ways to extract information from the rank of persistence modules (see [BOO21], [Lan14]). In this paper, we define an invariant on a persistence modules for an arbitrary finite poset using the rank of its internal morphisms. These *rank characters* assign to a persistence module a *virtual barcode*, which will agree with its actual barcode in the event that the module is a direct sum of convex modules. In addition, virtual barcodes provide a one-sided test for determining when a persistence module has only convex summands, in the sense that many virtual barcodes indicate that their corresponding module cannot be decomposed into a direct sum of only convex modules.

[1]In the literature, the corresponding modules are commonly called *interval modules.* We choose a different terminology to distinguish between actual intervals in the poset P and the supports of more general convex modules

$$\begin{tikzcd} & 4 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & 0 & \\ k \arrow[ur, "0"] & & k \arrow[ul, "0"'] \\ & k^2 \arrow[ul, "(1\ 0)"] \arrow[ur, "(1\ 0)"'] & \end{tikzcd} \qquad \begin{tikzcd} & 0 & \\ k \arrow[ur, "0"] & & k \arrow[ul, "0"'] \\ & k^2 \arrow[ul, "(0\ 1)"] \arrow[ur, "(1\ 0)"'] & \end{tikzcd}$$

$P \qquad\qquad M \qquad\qquad N$

FIGURE 1. The *diamond*, along with two representations that agree on every interval with respect to rank.

We'll now provide a brief background for rank characters; see Section 2 for complete definitions. If P is a poset, let $\Pi := \{S \subseteq P \mid S \text{ is convex}\}$. Each $S \in \Pi$ gives rise not only to a convex module M_S, but also an element A_S of the poset (incidence) algebra. We use the fact that a persistence module M can be applied to each A_S to obtain a linear operator on the vector space $\bigoplus_{p \in P} M(p)$.

The *rank character* of M is defined to be the $|\Pi| \times 1$ integer vector $\chi(M) := (\text{Rank}(M(A_S)))_{S \in \Pi}$, and the *rank matrix* for P is the $\Pi \times \Pi$ matrix $R = (r_{(S',S)}) := (\text{Rank}(M_S(A_{S'})))_{(S',S)}$. Our main results show how the rank character of a persistence module can be used to recover its barcode.

Main Theorem (Theorem 3.6)**.** *Let P be a finite, connected poset and let Π be the set of convex subsets of P.*

(a) *If M and N are direct sums of convex modules, then $\chi(M) = \chi(N) \iff M \cong N$.*
(b) *If M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ returns the barcode of M*

That is, rank characters are invariants for generalized persistence modules rich enough to distinguish between direct sums of convex modules. In this sense, they are analogous to the (ordinary) characters of a group. Moreover, the rank matrix is always invertible, and can be used to efficiently recover the isomorphism type of a direct sum of convex modules by matrix multiplication. This also provides a one-sided test which can sometimes be used to determine that a generalized persistence module *isn't* a direct sum of convex modules, since it follows easily from the Main Theorem that if $R^{-1} \cdot \chi(M)$ is not a vector over $\mathbb{Z}_{\geq 0}$, M is not a direct sum of convex modules. Thus, as an easy consequence of our main results we get the following corollary.

Corollary 1.2. *Let P be a finite, connected poset and M be an arbitrary generalized persistence module for P. Let R be the rank matrix for the set of convex subsets Π of P and $\chi(M)$ be the rank character of M. If $R^{-1} \cdot \chi(M)$ has a negative entry, then M is not a direct sum of convex modules.*

In Theorem 3.6, the vector $R^{-1} \cdot \chi(M) = (q_S)_{S \in \Pi}$, where each $q_S \in \mathbb{Q}$, is the *virtual barcode* of M. With motivation both from virtual characters for groups and [BOO21], our main results can be restated as follows.

Main Theorem (Theorem 3.8)**.** (Virtual barcode characterization) *Let P be a finite, connected poset and let M and N be persistence modules for P.*

(a) *If $M \cong N$, then the virtual barcodes of M and N are equal.*
(b) *If M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ is a vector over $\mathbb{Z}_{\geq 0}$, and $R^{-1} \cdot \chi(M) = (q_S)_{S \in \Pi}$ is the barcode of M.*
(c) *If $R^{-1} \cdot \chi(M)$ is not a vector over $\mathbb{Z}_{\geq 0}$, then M is not a direct sum of convex modules.*

That is, virtual barcodes are invariants for persistence modules which agree with barcodes when restricted to direct sums of convex modules. Moreover, they can also sometimes be used to detect if a module has non-convex summands. We remark that if the poset P has the property that its rank matrix is in $\text{SL}(|\Pi|, \mathbb{Z})$ as opposed to $\text{GL}(|\Pi|, \mathbb{Q}) \cap \text{M}_{|\Pi|,|\Pi|}(\mathbb{Z})$, the virtual barcode for every persistence module for this poset is a vector over $\mathbb{Z}$ (as opposed to $\mathbb{Q}$). In this situation, by interpreting the nonzero entries as multiplicities, the virtual barcode gives us a *virtual decomposition*

$$\begin{tikzcd}
\text{rep P} \arrow[r, "\chi(-)"] \arrow[dr, dashed] & \mathbb{Z}^{\Pi}_{\geq 0} \arrow[d, "R^{-1}\cdot -"] \\
& \mathbb{Q}^{\Pi}
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, mapsto] \arrow[dr, dashed, mapsto] & \chi(M) \arrow[d, mapsto] \\
& R^{-1}\cdot\chi(M)
\end{tikzcd}$$

FIGURE 2. A schematic for rank characters and virtual barcodes.

$$M \sim \bigoplus_{i=1}^{m} S_i - \bigoplus_{j=1}^{n} T_j$$

with $\chi(M) = \chi(\oplus S_i) - \chi(\oplus T_j))$. This decomposition is good up to isomorphism class, and agrees with the Krull-Schmidt decomposition when M is a direct sum of convex modules (in which case $n = 0$). We caution the reader, however, that, depending on the poset, many nonisomorphic persistence modules may have the same virtual barcode, even in the event that the barcode is over $\mathbb{Z}_{\geq 0}$; see Example 4.3.

This paper is organized as follows. In Section 2, we cover some preliminaries and formally define rank characters and the rank matrix. Section 3 contains our main results, while in Section 4 we provide examples of some explicit computations and also point out some of the subtleties of rank characters. Finally, in Section 5 we point out some questions raised by our work that we wish to consider in the future.

2. CONVEX MODULES AND RANK CHARACTERS

In this section we'll introduce the ideas that we will use throughout the remainder of the paper and prove some technical results which we will need moving forward.

Definition 2.1. *Let P be a poset. A nonempty subset S is called* ***poset convex*** *if for any $a, b \in S$, the set $[a, b] := \{p \in P \mid a \leq p \leq b\}$ is a subset of S. S is* ***convex*** *if it is poset convex and edge connected.*

It's well known that convex sets serve as the support for some special indecomposable persistence modules. If $S \subseteq P$ is convex, the module M_S defined below is a well-defined indecomposable persistence module for P.

$$M_S(p) := \begin{cases} k & \text{if } p \in S \\ 0 & \text{if } p \notin S \end{cases} \qquad \text{and} \qquad M_S(a \leq b) := \begin{cases} 1 & \text{if } \{a, b\} \subseteq S \\ 0 & \text{if } \{a, b\} \not\subseteq S. \end{cases}$$

If M is a persistence module for P isomorphic to some M_S, we say M is convex. While in the literature, these modules are commonly called *interval modules*, we choose different terminology to distinguish between actual intervals in the poset P and the supports of more general convex modules (see, for example, Figure 3). An indecomposable persistence module with the property that each vector space is either zero or one-dimensional must have support given by a convex set; such modules are commonly called *thin*. Convex sets also give rise to elements of the poset algebra. Let S be convex and set $A_S = \sum [a_i, b_i]$, where the sum is over the intervals in P maximal with respect to being contained in S. That is,

$$A_S := \sum_{[a_i, b_i] \in T_S} [a_i, b_i], \qquad T_S := \{[a, b] \mid [a, b] \subseteq S \text{ and } [a, b] \subseteq [c, d] \subseteq S \Rightarrow [a, b] = [c, d]\}.$$

Note that since we may view the poset algebra as the path algebra for the Hasse quiver of P modulo the parallel ideal, A_S is a well-defined element of the poset algebra. Let $\Pi := \{S \subseteq P \mid S \text{ is convex}\}$. We already know that Π parameterizes the collection of convex modules, and since every convex subset S is uniquely determined by its collection of maximal intervals, the assignment $S \mapsto A_S$ is one-to-one. So each element S of Π corresponds to both a convex module M_S and a unique element of the poset algebra A_S. In this section we'll be careful to distinguish between convex subsets and their corresponding convex modules, but in what follows when there is no ambiguity we will no longer make this distinction.

Definition 2.2. *Let P be a finite poset and V be an arbitrary persistence module for P.*

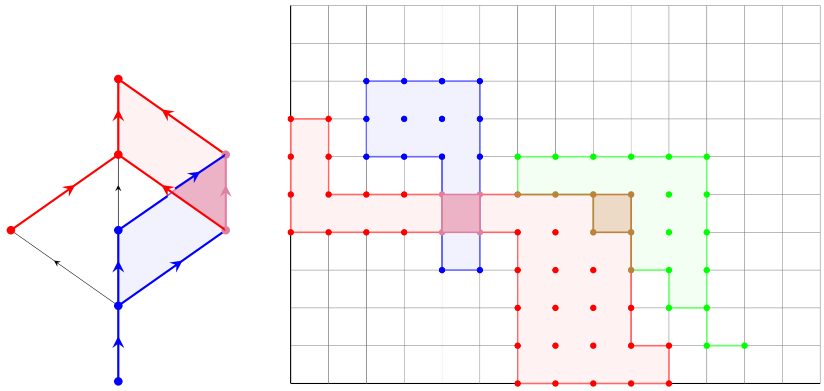


Figure 3. On the left, the red and blue sets are convex subsets for the pictured poset. On the right, the pink, blue and green modules are all convex subsets of the commutative grid.

(i) *The* ***rank character of*** V*, denoted* $\chi(V)$*, is the* $|\Pi| \times 1$ *column vector* $\big(\mathrm{Rank}(V(A_{M'}))\big)$*, where* $M' \in \Pi$.
(ii) *The* ***rank matrix*** *for* P *is the* $|\Pi| \times |\Pi|$ *matrix* $R = (r_{(S',S)}) = (\mathrm{Rank}(M_S(A_{S'})))_{(S',S)}$.

Thus, to compute the S'-entry of $\chi(V)$, we apply V to $A_{S'}$ and take the rank of the resulting matrix. Clearly, $\chi(V)$ has nonnegative integer coefficients. The rank matrix is the square matrix whose columns are the characters of the convex modules for P. (We always have some fixed ordering on Π.) We refer the reader to Examples 4.1-4.5 for some sample computations.

In the next section we'll show that rank matrix is invertible, and that rank characters are invariants for persistence modules rich enough to distinguish between direct sums of convex modules. We begin with some preliminary observations about convex subsets.

Lemma 2.3. *Let* S, S' *be poset convex, with* S' *given by the collection of maximal intervals* $T_{S'} = \{[a_i, b_i]\}_{i=1}^n$*. Let* $U_S \subseteq T_{S'}$ *be the collection of maximal intervals which are contained in* S*. Then* $\bigcup_{t \in U_S} t$ *is poset convex.*

Proof. Let $x, y \in \bigcup_{t\in U_S} t$, with $x \leq y$. Then $x \in [a_x, b_x]$ and $y \in [a_y, b_y]$, where $[a_x, b_x], [a_y, b_y] \in U_S$. Thus,

$$a_x \leq x \leq y \leq b_y,$$

so $[a_x, b_y] \subseteq S$, as $a_x, b_y \in S$. But $[a_x, b_y] \subseteq S'$, so $[a_x, b_y] \in U_S$. Therefore, $[x, y] \subseteq \bigcup_{t\in U_S} t$ as required. $\square$

We'll now define two subsets of $T_{S'}$. Let S' be poset convex. Let $\mathcal{G}(S') := \mathcal{P}(T_{S'})\backslash\{\emptyset\}$ and $\mathcal{F}(S') := \{\underline{S} \in \mathcal{G}(S') \mid \underline{S} \text{ is poset convex}\}$, where $\mathcal{P}(T)$ denotes the power set of a set T. Both sets are posets with ordering given by containment, and clearly $\mathcal{F}(S')$ is a subposet of $\mathcal{G}(S')$. Intuitively, $\mathcal{F}(S')$ corresponds to the lattice of thin submodules of S' which are the direct sums of convex modules having support given only by elements of $T_{S'}$. When S' is understood, we will sometimes write $\mathcal{F}$ and $\mathcal{G}$, respectively.

Definition 2.4. *Let* $R_1, R_2 \in \mathcal{F} = \mathcal{F}(S')$*. Denote*

$$I_{R_1}^{R_2} := \{S \in \Pi \mid \{t \in T_{R_2} \mid t \subseteq S\} = T_{R_1}\}.$$

Thus, $I_{R_1}^{R_2}$ will correspond to the set of convex modules whose fiber is the same, when S' is replaced by R_2. Because of this, we call the set $I_{R_1}^{R_2}$ the R_1 **fiber of** R_2. Clearly if $I_{R_1}^{R_2} \neq \emptyset$, then $R_2 \supseteq R_1$. Note that this notation makes sense for $R_1, R_2 \in \mathcal{G}$. In this situation, we have the following lemma.

Lemma 2.5. *Let* $R_1, R_2 \in \mathcal{G}$*,* $S' \in \mathcal{F}$*, and* $R_1 \leq R_2 \leq S'$*. Then* $I_{R_1}^{R_2} = \bigsqcup_{x\in\mathcal{P}(X)} I_{R_1\sqcup x}^{S'}$*, where* X *is a set defined by the property* $T_{R_2} \sqcup X = T_{S'}$ *and all the unions are disjoint. Moreover,* $I_{R_1\sqcup x}^{S'} \neq \emptyset$ *implies* $R_1 \sqcup x \in \mathcal{F}$.

Note that since $R_2 < S'$, X is uniquely defined.

Proof. Let $S \in I_{R_1}^{R_2}$, so $\{t \in T_{R_2} \mid t \subseteq S\} = T_{R_1}$. Thus, $S \supseteq [a,b]$ for every maximal interval $[a,b] \in T_{R_1}$. Now consider $A = \{t \in T_{S'} \mid t \subseteq S\}$. By inspection, $A \supseteq T_{R_1}$, hence $A = R_1 \sqcup x$ for some x and $A = T_S$. Thus, by Lemma 2.3, the union of intervals in A is poset convex, so A is in $\mathcal{F}$. The result holds because the nonempty sets $\{I_{R_1 \cup x}^{S'}\}_{x \in \mathcal{P}(X)}$ form a partition of $I_{R_1}^{R_2}$. □

The next lemma will give us some identities involving fibers which we'll use in the next section.

Lemma 2.6. *Fix S'. We have the following:*

(a) *If $S_1 < S_2 < S'$ with $S_1, S_2 \in \mathcal{F}$ and S_2 is maximal in $\mathcal{F}\backslash\{S'\}$, and if there is no $N \in \mathcal{F}$ such that $S_1 < N < S_2$, then*

$$I_{S_1}^{S_2} = I_{S_1}^{S'} \sqcup I_{S'}^{S'} \sqcup \bigsqcup I_{S_j}^{S'},$$

where $S_1 < S_j < S'$ are all in $\mathcal{F}$ and $S_j \neq S_2$.

(b) *If $S_1 < S'$ and $S_1 \in \mathcal{F}$, then $I_{S_1}^{S_1} = \bigsqcup_{S \in \mathcal{F}, S_1 \leq S} I_S^{S'}$.*

Proof. The first statement follows from Lemma 2.5 by identifying $S_1 = R_1$, $S_2 = R_2$, and considering only points in $\mathcal{F}$. The second statement is just a special case of the first, where $R_1 = R_2 = S_1$. □

These properties of fibers will be used in the proof of our main results in the next section.

3. Main Results

Throughout this section, fix two modules L_1 and M_1, where $L_1 = \bigoplus_{L \in \mathcal{S}} \bigoplus_{i_L=1}^{m_L} L$ and $N_1 = \bigoplus_{N \in \mathcal{T}} \bigoplus_{i_N=1}^{n_N} N$, where $\mathcal{S}, \mathcal{T} \subseteq \Pi$.

Lemma 3.1. *Let $L_1 = \bigoplus_{L \in \mathcal{S}} \bigoplus_{i_L=1}^{m_L} L$ and $N_1 = \bigoplus_{N \in \mathcal{T}} \bigoplus_{i_N=1}^{n_N} N$ be as above. If $\chi(L_1) = \chi(N_1)$, then L_1 and N_1 agree on all elements of $\mathcal{F}$.*

That is, if two direct sums of convex modules agree on A_R for all R convex, then they agree on A_R for $R \in \mathcal{F}$ as well.

Proof. Let $M' \in \mathcal{F}$ and suppose $M' = M_2 \oplus M_3$ for M_2 and M_3 convex. Then, since the module M' is thin, M_2 and M_3 must partition both the upper and lower endpoints of each maximal interval in $T_{M'}$. So for each M convex, the matrix for $M(A_{M'})$ must be block diagonal:

$$M(A_{M'}) = \begin{pmatrix} M(A_{M_2}) & 0 \\ 0 & M(A_{M_3}) \end{pmatrix}$$

Thus, $\mathrm{Rank}(M(A_{M'})) = \mathrm{Rank}(M(A_{M_2})) + \mathrm{Rank}(M(A_{M_3}))$, and the result follows. □

The following proposition will establish equivalent conditions for the rank matrix being nonsingular. See the next section for examples illustrating how to compute the rank matrix for an explicit poset, specifically, Examples 4.1 and 4.5.

Proposition 3.2. *Let P be a poset, and let R be the rank matrix for P. Then R is singular if and only if there exist two disjoint nonempty subsets of Π, labeled A and B, and two direct sums of convex modules, M_A and M_B, supported on A and B, respectively, with $\chi(M_A) = \chi(M_B)$.*

Proof. Let Π be as above, and consider the vector $(m_M) \in \mathbb{Z}_{\geq 0}^{|\Pi|}$. Then,

$$\begin{aligned} R \cdot (m_M) &= (\mathrm{Rank}(M(A_{M'}))) \cdot (m_M) \\ &= \left(\sum_{M \in \Pi} m_M \cdot \mathrm{Rank}(M(A_{M'})) \right) \\ &= \left(\sum_{M \in \Pi} \mathrm{Rank} \left(\left(\bigoplus_{i_M=1}^{m_M} M \right) (A_{M'}) \right) \right) \\ &= \left(\mathrm{Rank} \left(\left(\bigoplus_{M} \bigoplus_{i_M=1}^{m_M} M \right) (A_{M'}) \right) \right) \\ &= \chi \left(\bigoplus_{M \in \Pi} \bigoplus_{i_M=1}^{m_M} M \right). \end{aligned} \tag{1}$$

Thus, left multiplication by R sends the barcode of a direct sum of convex modules to its character. Therefore,

$$\begin{aligned} R \text{ is singular} &\iff \text{there exists a rational linear dependence among the columns of } R \\ &\iff \text{there exists a positive rational linear dependence among the columns of } R \\ &\iff \text{there exists a positive integral linear dependence among the columns of } R \\ &\iff \left(\sum_{M \in A} m_M \cdot \mathrm{Rank}(M(A_{M'})) \right) = \left(\sum_{N \in B} m_N \cdot \mathrm{Rank}(N(A_{M'})) \right), \end{aligned}$$

where A and B are two disjoint subsets of Π, and m_M and m_N are positive integers. Therefore, the result is proved by (1). $\square$

The following corollary follows immediately from Equation (1) once R is nonsingular.

Corollary 3.3. *If the rank matrix R is nonsingular and M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ returns the barcode of M.*

We are now ready to prove the main result establishing that rank characters separate points for direct sums of convex modules.

Theorem 3.4. *Suppose L and N are two direct sums of convex modules with $\chi(L) = \chi(N)$. Then $L \cong N$.*

Proof. If the theorem is false, then by Proposition 3.2 there exist $L_1 = \bigoplus_{L \in \mathcal{S}} \bigoplus_{i_L=1}^{m_L} L$ and $N_1 = \bigoplus_{N \in \mathcal{T}} \bigoplus_{i_N=1}^{n_N} N$, where $\mathcal{S}, \mathcal{T} \subseteq \Pi$, with $\mathcal{S} \cap \mathcal{T} = \emptyset$ and $\chi(L_1) = \chi(N_1)$. For M_1 and $M_2 \in \mathcal{F}$, let $K_{M_1}^{M_2}$ and $J_{M_1}^{M_2}$ be the collections of summands of L_1 and N_1, respectively, whose isomorphism classes lie in $I_{M_1}^{M_2}$ (recall Definition 2.4).

We'll show that if this is the case, then for any $M' \in \mathcal{F}(P)$ and for any $\underline{M} \in \mathcal{F}(M')$, we have $|K_{\underline{M}}^{M'}| = |J_{\underline{M}}^{M'}|$. We proceed by induction on the cardinality of $T_{M'}$. If $|T_{M'}| = 1$ then $M' = [a, b]$ for some a and b, so

$$\mathrm{Rank}(M(A_{M'})) = \begin{cases} 1 & \text{if } [a,b] \subseteq M \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\left(\sum_{L \in \mathcal{S}} m_L \cdot \mathrm{Rank}(L(A_{M'})) \right)$ is $|K_{M'}^{M'}|$. Since $\mathcal{F}(M') = \{M'\}$ and $\left(\sum_{N \in T} n_N \cdot \mathrm{Rank}(N(A_{M'})) \right) = |J_{M'}^{M'}|$, the base case holds.

Now let $|T_{M'}| > 1$ and suppose the result holds for all elements of $\mathcal{F}(P)$ with a smaller collection of maximal intervals. That is, suppose the result holds for all $M'' \in \mathcal{F}(P)$ with $|T_{M''}| < |T_{M'}|$. Under these hypotheses, we obtain the following fact.

Fact. *For every $M \in \mathcal{F}(M')$, we have $|K_M^{M'}| - |J_M^{M'}| \in \mathbb{Z}(|K_{M'}^{M'}| - |J_{M'}^{M'}|)$. In particular, if $\underline{M} \leq M$ and $\underline{M}$ is maximal in $\mathcal{F}(M')\backslash\{M'\}$, then*

$$|K_{\underline{M}}^{M'}| - |J_{\underline{M}}^{M'}| = -(|K_{M'}^{M'}| - |J_{M'}^{M'}|) \tag{2}$$

Proof of Fact. First, suppose $\underline{M}$ is maximal in $\mathcal{F}(M')\backslash\{M'\}$. Then, by the induction hypothesis, $|K_{\underline{M}}^{\underline{M}}| = |J_{\underline{M}}^{\underline{M}}|$. But by Lemma 2.6, $K_{\underline{M}}^{\underline{M}} = K_{\underline{M}}^{M'} \sqcup K_{M'}^{M'}$ and $J_{\underline{M}}^{\underline{M}} = J_{\underline{M}}^{M'} \sqcup J_{M'}^{M'}$, proving (2). By reverse induction and repeated use of Lemma 2.6, the general statement follows.[2] □

With this Fact proved, we can continue with the proof of the theorem. Now suppose that there is some $M_1 \in \mathcal{F}(M')$ with the property that every chain of inclusions from M_1 to M' in $\mathcal{F}(M')$ has length exactly two. That is, every chain of proper inclusions $M_1 < \underline{M} < M'$ in $\mathcal{F}(M')$ cannot be lengthened, and M_1 is not maximal in $\mathcal{F}(M')\backslash\{M'\}$. When this is the case, we say informally that M_1 is *two from the top*.

Let M_1 be two from the top in $\mathcal{F}(M')$. Now, $|K_{M_1}^{M_1}| = |J_{M_1}^{M_1}|$ by the induction hypothesis. Rewriting this by using Lemma 2.6, we have

$$|K_{M_1}^{M'}| + \sum_{M_1 < \underline{M} < M'} |K_{\underline{M}}^{M'}| + |K_{M'}^{M'}| = |J_{M_1}^{M'}| + \sum_{M_1 < \underline{M} < M'} |J_{\underline{M}}^{M'}| + |J_{M'}^{M'}|.$$

Hence,

$$0 = |K_{M_1}^{M'}| - |J_{M_1}^{M'}| + \sum_{M_1 < \underline{M} < M'} (|K_{\underline{M}}^{M'}| - |J_{\underline{M}}^{M'}|) + |K_{M'}^{M'}| - |J_{M'}^{M'}| = |K_{M_1}^{M'}| - |J_{M_1}^{M'}| + (1-k)(|K_{M'}^{M'}| - |J_{M'}^{M'}|)$$

where k is some integer by Equation (2) in the Fact. Also, when M_1 is maximal in $\mathcal{F}(M_2)\backslash\{M_2\}$, we have

$$|K_{M_1}^{M'}| - |J_{M_1}^{M'}| + \sum_{M_1 < \underline{M} < M'} (K_{\underline{M}}^{M'} - |J_{\underline{M}}^{M'}|) + |K_{M'}^{M'}| - |J_{M'}^{M'}| = 0$$

$$\Rightarrow \; |K_{M_1}^{M'}| - |J_{M_1}^{M'}| + (1-l)(|K_{M'}^{M'}| - |J_{M'}^{M'}|) = 0$$

for some integer l, which is necessarily less than k since M_2 is not in this summation. Thus, $|K_{M'}^{M'}| - |J_{M'}^{M'}| = 0$. Then, by the Fact, we also have $|K_{\underline{M}}^{M'}| - |J_{\underline{M}}^{M'}| = 0$ for all $\underline{M} \in \mathcal{F}$. Therefore, if the lattice for $\mathcal{F}(M')$ has height at least two, our proof by induction is complete since a finite poset of height at least two with a unique maximum element must have an element that is two from the top. On the other hand, if $|S_{M'}| > 1$ and the lattice for $\mathcal{F}(M')$ doesn't have height at least two, then M' is a direct sum of pairwise disjoint intervals. In this case the inductive proof is done as well by Lemma 3.1.

Therefore, we have shown that if $\chi(L_1) = \chi(N_1)$, then $|K_{\underline{M}}^{M'}| = |J_{\underline{M}}^{M'}|$ for all $M' \in \mathcal{F}(P)$ with $\underline{M} \leq M'$. Thus, since $\mathcal{S} \cap \mathcal{T} = \emptyset$, without loss of generality, we may pick $N' \in \mathcal{T}$ with $N' \not\subseteq L'$ for any $L' \in \mathcal{S}$. Then, $|J_{N'}^{N'}| > 0$ but $|K_{N'}^{N'}| = 0$, a contradiction. Thus, Theorem 3.4 is proved. □

As a consequence of this technical result, we have the following useful proposition, which will next allow us to connect rank characters with barcodes.

Proposition 3.5. *Let P be a finite, connected poset and let Π be the set of convex subsets of P. Then the rank matrix $R = (\mathrm{Rank}(M_S(A_{S'})))_{(S,S') \in \Pi^2}$ is invertible.*

Proof. Let P be finite and connected. By Theorem 3.4 and Proposition 3.2, the rank matrix R for P is invertible since its columns are linearly independent. □

We are now ready to prove the two main theorems stated in the introduction.

Theorem 3.6. *Let P be a finite, connected poset and let Π be the set of convex subsets of P.*

(a) *If M and N are direct sums of convex modules, then $\chi(M) = \chi(N) \iff M \cong N$.*

[2] We remark that we never used the assumption that $\mathcal{S}$ and $\mathcal{T}$ are disjoint in the proof of this statement. Thus, in a sense, we first prove that modules with the same character satisfy the conclusion of the statement.

(b) *If M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ returns the barcode of M*

Proof. It's clear that for any persistence modules M and N, if $M \cong N$, then $\chi(M) = \chi(N)$. Then, by the computation in Equation (1) and by Corollary 3.3, the result immediately follows. □

With motivation from this theorem, we make the following definition.

Definition 3.7. *Let M be a persistence module for P. The vector $R^{-1} \cdot \chi(M) = (q_S)_{S \in \Pi}$, with each $q_S \in \mathbb{Q}$, is the* ***virtual barcode of*** *M.*

By Proposition 3.5, the virtual barcode of a persistence module M is an invariant which agrees with the Krull-Schmidt decomposition of M in the event that it is a direct sum of convex modules. When this is the case, the barcode is recovered simply by matrix multiplication. The virtual barcode can also sometimes be used to detect when a module has non-convex summands, since negative or properly rational entries show that a module is not *purely convex*. These results are formulated in the next theorem.

Theorem 3.8. (Virtual barcode characterization) *Let P be a finite, connected poset and let M and N be persistence modules for P.*

(a) *If $M \cong N$, then the virtual barcodes of M and N are equal.*
(b) *If M is a direct sum of convex modules, then $R^{-1} \cdot \chi(M)$ is a vector over $\mathbb{Z}_{\geq 0}$, and $R^{-1} \cdot \chi(M) = (q_S)_{S \in \Pi}$ is the barcode of M.*
(c) *If $R^{-1} \cdot \chi(M)$ is not a vector over $\mathbb{Z}_{\geq 0}$, then M is not a direct sum of convex modules.*

Proof. Part (a) is a restatement of Theorem 3.6(a). Parts (b) and (c) also easily follow from Theorem 3.6 by contraposition. Note that part (c) is a strengthening of Corollary 1.2. □

Thus, any persistence module M has a rank character $\chi(M)$, which is a nonnegative integer vector indexed by Π. This rank character is an invariant for M, which can be used to associate to M its virtual barcode $R^{-1} \cdot \chi(M)$. If M is a direct sum of convex modules, its virtual barcode is precisely its barcode, and if its virtual barcode has an entry not in $\mathbb{Z}_{\geq 0}$, M must have non-convex summands. We remind the reader that the hypothesis that M and N are convex in Theorem 3.6 is necessary (see Example 4.3). If a module's virtual barcode is in $\mathbb{Z}_{\geq 0}$ this only says that there is a direct sum of convex modules with the same rank character (see Example 4.4). For a general persistence module, by separating the positive and negative entries of the virtual barcode, we get the association

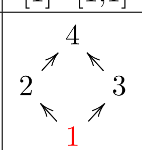
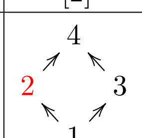
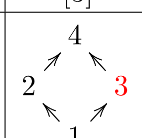
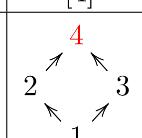
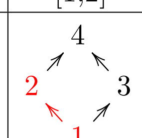
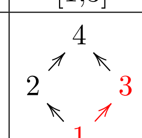
$$M \sim \bigoplus_{i=1}^{m} q_i S_i - \bigoplus_{j=1}^{n} r_j T_j$$

where the multiplicities q_i and r_j are possibly rational[3].

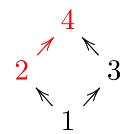
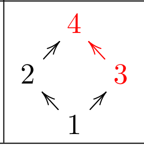
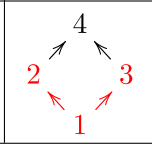
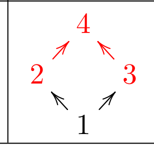
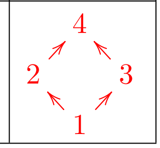
4. Examples

In this section, we provide some examples of explicit computations with rank characters.

Example 4.1. We return to the diamond (see Example 1.1 and Figure 1). Below is a table for $\Pi = \Pi(P)$ with a chosen ordering:

#	1	2	3	4	5	6
S	{1}	{2}	{3}	{4}	{1,2}	{1,3}
A_S	[1]=[1,1]	[2]	[3]	[4]	[1,2]	[1,3]
Pictorial depiction						

[3]For every poset P that we've computed, every persistence module has an integral virtual barcode. At this point, we do not know if this is true in general.

#	7	8	9	10	11
S	{2,4}	{3,4}	{1,2,3}	{2,3,4}	{1,2,3,4}
A_S	[2,4]	[3,4]	[1,2]+[1,3]	[2,4]+[3,4]	[1,4]
Pictorial depiction					

Then, the rank matrix $R = (\mathrm{Rank}(M_S(A_{S'})))_{S,S'\in\Pi}$ is [4]

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 12 \\ 13 \\ 24 \\ 34 \\ 123 \\ 234 \\ 1234
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 12 & 13 & 24 & 34 & 123 & 234 & 1234
\end{array} \\
\left(\begin{array}{cccc|cccc|cc|c}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

The characters $\chi(M)$ and $\chi(N)$ are given below, along with their virtual barcodes. By Theorem 3.6, these are their barcodes as well.

$$\chi(M) = \left(2\,1\,1\,0\,1\,1\,0\,0\,1\,0\,0\right)^T \Rightarrow R^{-1}\cdot\chi(M) = \left(1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\right)^T$$
$$\chi(N) = \left(2\,1\,1\,0\,1\,1\,0\,0\,2\,0\,0\right)^T \Rightarrow R^{-1}\cdot\chi(N) = \left(0\,0\,0\,0\,1\,1\,0\,0\,0\,0\,0\right)^T$$

Thus, the decompositions of M and N are determined to be

$$M \cong M_{\{1\}} \oplus M_{\{1,2,3\}} \text{ and } N \cong M_{\{1,2\}} \oplus M_{\{1,3\}}.$$

Recall that the rank of internal linear transformations alone isn't sufficient to distinguish between M and N. $\square$

Example 4.2. When P is totally ordered, convex subsets correspond *exactly* to intervals in P. In this situation, P is given by the equioriented A_n quiver.

$$A_n : 1 \to 2 \to \cdots \to n-1 \to n$$

In this situation the (b,a) entry of $M_S(A_{S'})$ is given by the identity:

$$M_{[x,y]}(A_{[z,w]})_{b,a} = \begin{cases} 1 & \text{if } x \leq a = z \leq b = w \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{Rank}(M_{[x,y]}(A_{[z,w]})) = \begin{cases} 1 & \text{if } x \leq z \leq w \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, when P is totally ordered, in the standard ordering the rank matrix will be upper triangular with ones on the main diagonal (and hence is clearly invertible). $\square$

[4] Blocks are added to the rank matrix only for readability. For details regarding the computation of the entries, see Example 4.5.

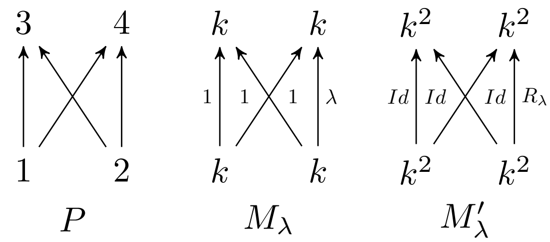


Figure 4. P with two representations that depend on λ.

Example 4.3. In this example we show that while two persistence modules may have the same character (and hence the same virtual barcode) at most one can be a direct sum of convex modules. Consider the poset P in Figure 4 and persistence modules M_λ and M'_λ, where $\lambda \in k$ and $R_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$.

Then M_λ is an indecomposable thin module for any λ, but is a direct sum of convex modules only when λ is 1 (when it is convex). On the other hand, $\chi(M_\lambda) = \chi(M_1)$ for all $\lambda \neq 0$. Thus, the rank character (and rank in general) cannot distinguish between M_1 and M_λ for $\lambda \neq 0$. Similarly, M'_λ is indecomposable whenever $\lambda \neq 0$, yet M'_λ is not a direct sum of convex modules except in the case $\lambda = 0$, in which case $M'_0 \cong M_1 \oplus M_1$. □

Example 4.4. In this example we show how rank characters can sometimes be used to conclude that a persistence module is *not* a direct sum of convex modules (see Corollary 1.2 and Theorem 3.8). Consider the poset P and module M given in Figure 5.

One can check that the rank matrix $R = (\mathrm{Rank}(M_S(A_{S'}))_{(S,S')}$ is as below.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 14 \\ 24 \\ 34 \\ 124 \\ 134 \\ 234 \\ 1234 \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 14 & 24 & 34 & 124 & 134 & 234 & 1234 \end{array} \\
\left(\begin{array}{cccc|ccc|ccc|c}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) \end{array}$$

By direct computation, $\chi(M) = (1\,1\,1\,2\,1\,1\,1\,2\,2\,2\,2)^T$ and $R^{-1} \cdot \chi(M) = (0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,{-1})^T$. Thus, since the virtual barcode has a negative coordinate, by Theorem 3.8, M cannot be a direct sum of convex modules. Alternatively, the virtual barcode suggests the association

$$M \sim M_{\{1,2,4\}} \oplus M_{\{1,3,4\}} \oplus M_{\{2,3,4\}} \ominus M_{\{1,2,3,4\}},$$

where

$$\chi(M) = \chi(M_{\{1,2,4\}} \oplus M_{\{1,3,4\}} \oplus M_{\{2,3,4\}}) - \chi(M_{\{1,2,3,4\}}).$$

From this viewpoint, as a convex module appears with negative multiplicity, M is not a direct sum of convex modules. The persistence modules in Example 4.3 also illustrate this phenomenon. For example, the module M_0 (see Figure 4) has the property that $R^{-1} \cdot \chi(M_0) = (0\,0\,0\,0\,{-1}\,0\,0\,0\,1\,0\,1\,0\,0)^T$, so M_0 is not a direct sum of convex modules (though it is indecomposable and thin) as $M_0 \sim M_{\{1,2,3\}} \oplus M_{\{1,3,4\}} \ominus M_{\{1,3\}}$.

Note that N is a direct sum of two convex modules, and its decomposition is recovered, since

$$\chi(N) = (1\,1\,1\,2\,1\,1\,1\,1\,2\,2\,2)^T \text{ and } R^{-1} \cdot \chi(N) = (0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,0)^T$$

Thus, $N \cong M_{\{3,4\}} \oplus M_{\{1,2,4\}}$ □

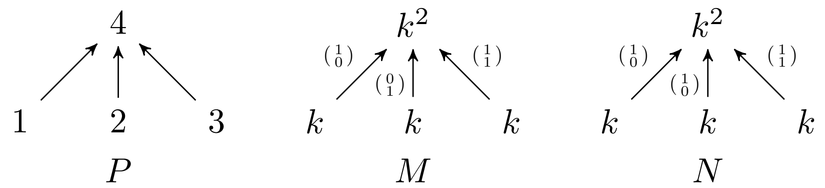


FIGURE 5. P with two representations.

Example 4.5. In this example we compute some entries of the rank matrix in Example 4.1, and then some entries of the characters in Example 4.4.

First, for the rank matrix in Example 4.1, consider $S = \{1,2,3\}$ and $S' = \{1,2,3,4\}$ for the diamond (see Figure 1). Then we have

$$M_S(A_S) = \begin{array}{c} \\ 1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad M_S(A_{S'}) = \begin{array}{c} \\ 1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Hence, $\mathrm{Rank}(M_S(A_S)) = 1$ and $\mathrm{Rank}(M_S(A_{S'})) = 0$, so the $(9,9)$ entry in the matrix is 1 and $(11,9)$ entry is 0.

Now for the characters $\chi(M)$ and $\chi(N)$ from Example 4.4. Since both modules have a two-dimensional vector space at vertex 4, the matrices will be 5×5. We then compute

$$M(A_{\{1,2,4\}}) = \begin{array}{c} \\ 1\\2\\3\\4\\4' \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 4' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad N(A_{\{1,2,4\}}) = \begin{array}{c} \\ 1\\2\\3\\4\\4' \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 4' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Hence, $\mathrm{Rank}(M(A_{\{1,2,4\}})) = 2$ while $\mathrm{Rank}(N(A_{\{1,2,4\}})) = 1$, which are precisely the ninth entries in the characters. □

Example 4.6. In this example we point out a combinatorial consequence of the rank matrix being invertible. This result is independent of our analysis of rank characters and virtual barcodes.

Let Π' be the set of all convex subsets of P which don't correspond to simples or arrows in the Hasse quiver. That is, $\Pi' = \{S \in \Pi \mid |S| > 2\}$. For $S, S' \in \Pi'$, let $d_{S,S'}$ be the number of arrows contained in S which are maximal intervals in S'. Then, the square matrix $(D_{S,S'})_{(S,S')\in\Pi'^2}$ is invertible, where

$$D_{S,S'} = \mathrm{Rank}(M_S(A_{S'})) - d_{S,S'}.$$

To see this, choose an ordering on Π so that all simples are first and all arrows are next. We first perform column operations corresponding to simples to clear out their rows in the rank matrix. Then we perform row operations corresponding to arrows to clear out their columns, obtaining a block diagonal matrix. The result follows, since the block corresponding to $\Pi' \times \Pi'$ must be invertible. □

5. FUTURE DIRECTIONS

The results of this paper raise several interesting questions which we wish to consider in future work. First, when does the rank matrix have determinant ± 1?[5] This is relevant, since any such poset will have the property that *every* persistence module has a virtual barcode in $\mathbb{Z}$. As we have seen, when this is the

[5]My thesis student at Reed College, Oliver Mansbach, has shown that this need not always happen even for subposets of $\mathbb{Z} \times \mathbb{Z}$.

case, every persistence module has the same rank character as a difference of two direct sums of convex modules.

Second, we have seen that if the virtual barcode of a persistence module N is in $\mathbb{Z}_{\geq 0}$, there is a unique direct sum of convex modules M with $\chi(N) = \chi(M)$. With motivation from representation theory, the authors wonder whether the inverse image of a rank character can be finite without being a singleton. That is, if k is an infinite field, and $|\{M \mid \chi(M) = \chi(N)\}| > 1$, must the set necessarily be infinite? (We can show this to be false for a finite field k.) In more generality, what can one say about the inverse image of the function $M \to \chi(M)$? For example, each level set consists of persistence modules with the same dimension vector. What other common properties must modules with the same character share?

Lastly, the authors wonder if orthogonality relations analogous to those for group characters exist. That is, is there an inner product of the set of characters of convex modules that has a nice physical interpretation?

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